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Sound radiation from turbulent boundary layers formed on compliant surfaces

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The paper considers the effect of turbulence-induced surface response on the sound radiated by a turbulent boundary layer. The analysis is confined to an infinite plane homogeneous surface and the conclusions may not be a good indication of the behaviour of more realistic structures. The main result of the analysis is that no fundamentally more efficient source of sound is introduced by the surface motion. The radiation remains quadrupole in character. The surface merely accounts for a reflexion of the turbulence-generated sound, with the reflexion coefficient being identical to that of plane acoustic waves. Dissipation in the surface reduces the magnitude of the image system. A brief discussion of the effect on the particular quadrupoles to be found in a turbulent boundary layer concludes the paper. There it is argued that the radiation will probably be increased by surface motion, but not by an order of magnitude.

1. Introduction

The sound radiated by turbulent boundary layers can be affected in two distinct ways by any motion of the bounding surface. The most obvious possibility is that the surface should act as a sounding board excited by the turbulent pressure field, a situation likely to result in a substantial increase in the radiated sound. The response velocities would usually remain small in comparison with those of the turbulent flow, but the motion is potentially so much more effective as an acoustic radiator that it could well overshadow the quadrupoles acoustically equivalent to the boundary-layer turbulence. The other possibility would arise if the surface response induced a change in the turbulence structure. Then the surface could again act as a sounding board, but would be excited by a turbulent pressure field modified by the boundary motion in a rather unpredictable way. This paper ignores that aspect and deals only with the influence of a compliant surface on the radiation field of turbulence that remains unaffected by surface response. The response velocity is therefore considered to be much smaller than that characteristic of the turbulence, which is assumed known throughout the flow.

The new results that are presented seem quite contrary to dimensional arguments common to the literature on sound radiation from structures excited by turbulent flows (e.g. Lyamshev 1961). Those arguments show how surface motion, equivalent to a simple source distribution, induces a sound pressure that increases in direct proportion to a typical dynamic head of the boundary-layer flow. That source, being a fundamentally more efficient radiator than dipoles and quadrupoles acoustically equivalent to a rigid surface near turbulent flows, would seem the dominant contributor to any sound field generated by the interaction of turbulence with a compliant surface. For the same reason dipoles on rigid surfaces overwhelm any quadrupoles, because of the fundamental difference in their radiation efficiencies. It is now well known that that argument is sometimes misleading; the dipoles tend to do little but reflect the turbulence-generated sound on a plane rigid boundary and account for no fundamentally more efficient type of acoustic radiation. In a similar way, it is argued in this paper that surface motion, in the particular case of a large plane homogeneous structure, does not fundamentally alter the sound from that radiated by the turbulence in free space. In fact, the effect of the surface is shown to be accounted for by a classical reflexion coefficient, the responding surface merely accounting for the reflexion of a fraction (less than or equal to one) of turbulence-generated sound, but with a change in phase. Non-linear terms in response velocity, together with viscous effects, distort this property, but nevertheless the surface essentially maintains a passive role in the radiation problem. A similar but slightly different result holds for the non-propagating near-field.

The analysis is based on Powell's (1960) method of manipulating the aerodynamic noise theory due to Lighthill (1952) and Curle (1955). His approach is followed, almost exactly, to show how a 'pressure-release' surface accounts for the reflexion of an image of opposite phase and that the surface motion in that instance is not associated with any fundamentally more efficient source of sound. More general surfaces are treated by an approach which does not appear to have been used before in aerodynamic noise theory. The equations are decomposed into their spectral form, and the pressures related to particle velocities through impedance functions. This technique provides three simple simultaneous equations from which the effects of surface pressure and response velocity can be eliminated to yield a specific value for the radiation field. The paper is concluded with a brief discussion of the influence of surface properties on the sound radiated by the various quadrupole terms in boundary-layer flows.

2. Radiation from boundary layers on compliant surfaces

Whenever a surface is caused to vibrate, the fluid in its vicinity is set in motion and subjected to forces which radiate sound. If the vibration is of low amplitude, both the surface velocity and fluid stresses can be assumed to occur at the mean surface position. The radiated sound can then be estimated by well-established techniques where the compliant surface is replaced by a fixed control surface on which the velocity and stresses are set equal to those on the real surface. This procedure can be followed in studies of the sound radiated by turbulent flow established on plane compliant surfaces, particularly when the surface motion is assumed sufficiently small that it leaves unaltered the turbulence structure. That is the problem considered here, a problem that is pertinent to the mechanism of sound generation by turbulent boundary layers formed on flexible structures.

The equations governing the radiation were written down by Curle (1955) who showed, in his extension of Lighthill's (1952) theory of aerodynamic sound, how surface sources should augment quadrupoles that are acoustically equivalent to the turbulent volume. Curle's equation for the radiation field may be written in terms of the fluctuating pressure at a point $p(\mathbf{x},t)$

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_V [T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_s [\rho v_i v_n - P_i]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}. \tag{2.1}$$

T_{ij} is Lighthill's turbulence stress tensor, r is the distance separating the source point $\mathbf{y}$ from the observation point $\mathbf{x}$ ($r = |\mathbf{x}-\mathbf{y}|$), P_i is the force exerted on the fluid in the x_i direction by unit area of the boundary surface s, and v_i is a velocity component. Repeated tensor suffices are to be summed over 1, 2, and 3. The suffix n is not to be summed, it merely implies the component to lie in the direction of the outward normal from the volume V bounded by the surface s. The brackets [ ] indicate that the function they enclose is to be evaluated at the source position $\mathbf{y}$ at the retarded time $t - r/a_0$, a_0 being the speed of sound in the uniform medium where $\mathbf{x}$ is situated. That point must lie within the surface s, otherwise the pressure vanishes identically. This feature is a direct consequence of Kirchhoff's theorem that sources enclosed by a surface are fully equivalent to a source distribution on that surface, so that, if the surface separates the observation point $(\mathbf{x}, t)$ from the turbulence, no sound is heard. It was this property that enabled Powell (1960) to demonstrate how, for a plane surface, the surface terms accounted for little more than a reflexion of the sound generated in the turbulent volume. Powell's argument forms the basis of this analysis into the influence of surface motion on the radiated sound and is repeated here for completeness.

The real flow containing the observation point $(\mathbf{x}, t)$ lies above the surface s that lies at the mean position of the plane surface, a situation illustrated in figure 1. In that flow, a volume $V+$ is bounded by a closed surface $s+$ which is, in part, coincident with the surface s and, in part, sufficiently distant from the turbulence that sound has yet to reach it. On the other side of the surface s lies a hypothetical image system, a volume $V-$ bounded by the closed surface $s-$. The observation point $(\mathbf{x}, t)$ is excluded from the image system so that the net effect there of sources distributed throughout the volume $V-$ and the surface $s-$ is precisely zero. This point, as Powell remarks, makes it irrelevant that the image system is not physically realizable.

Curle's equation, when applied separately to the real and image system reproduces Powell's result

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+} [T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{s+} [\rho v_i v_n - P_i]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_{s+} \frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}, \tag{2.2}$$

$$0 = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-} [T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{s-} [\rho v_i v_n - P_i]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_{s-} \frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}. \tag{2.3}$$

Outside the turbulent flow, the fluid is assumed to be at rest and uniform with the pressure obeying the homogeneous wave equation.

The surface integrals vanish whenever the surface is in such a régime. In the problem of boundary-layer noise the surface integrals over $s+$ and $s-$ can be considered to vanish at all points where those surfaces do not coincide with s, the mean position of the boundary surface subjected to the turbulent-boundary-layer loading. Several elements in the integrals of equations (2.2) and (2.3) are

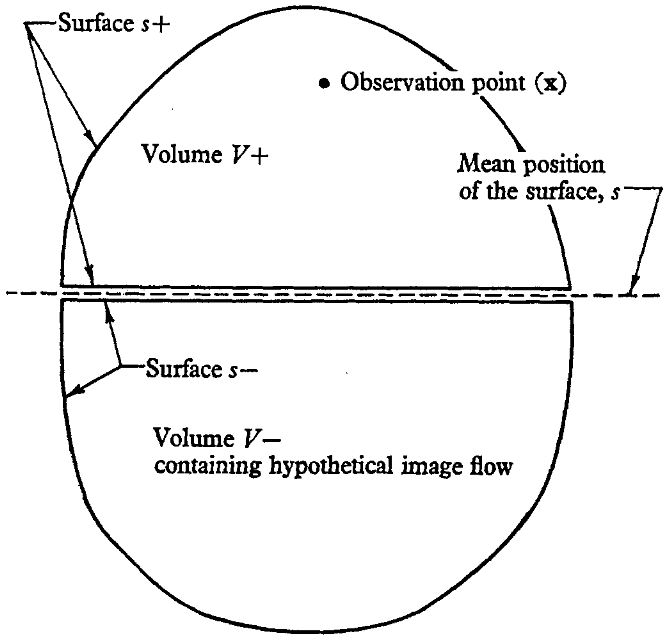

FIGURE 1. Diagram illustrating the reflexion properties of a plane surface subjected to turbulent flow.

then identical, a point that Powell demonstrated by breaking down the solution to those terms symmetric and those antisymmetric about the surface s. The symmetric terms can be assumed to act on the mean surface s and are the following:

$$\frac{1}{4\pi}\int_{s+}\frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r} = \frac{1}{4\pi}\int_{s-}\frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r} = \frac{1}{4\pi}\int_{s}\frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}, \tag{2.4}$$

and

$$\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int_{s+}[\rho v_\alpha v_n - P_\alpha]\frac{d\mathbf{y}}{r} = \frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int_{s-}[\rho v_\alpha v_n - P_\alpha]\frac{d\mathbf{y}}{r} = \frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int_{s}[\rho v_\alpha v_n - P_\alpha]\frac{d\mathbf{y}}{r}, \tag{2.5}$$

where α is a tensor suffix implying only those directions that lie in the surface s.

The antisymmetric term is

$$\frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_{s+}[\rho v_n^2 - P_n]\frac{d\mathbf{y}}{r} = -\frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_{s-}[\rho v_n^2 - P_n]\frac{d\mathbf{y}}{r}. \tag{2.6}$$

These relations simplify interpretation of equations (2.2) and (2.3) which can now be rewritten

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+}[T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_{s+}[\rho v_n^2 - P_n]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int_s [\rho v_\alpha v_n - P_\alpha]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}, \tag{2.7}$$

$$0 = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_{s+}[\rho v_n^2 - P_n]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int_s [\rho v_\alpha v_n - P_\alpha]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \frac{\partial}{\partial t}[\rho v_n]\frac{d\mathbf{y}}{r}. \tag{2.8}$$

Addition of these two equations shows how, on a rigid surface, the normal stresses account for reflexion that is distorted only by viscous terms. We shall return to this point, but for the present we are more concerned with the effects of surface response. One such effect is obvious at this stage, and follows by a subtraction of equation (2.8) from equation (2.7), giving

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+}[T_{ij}]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{2\pi}\frac{\partial}{\partial x_n}\int_{s+}[\rho v_n^2 - P_n]\frac{d\mathbf{y}}{r}. \tag{2.9}$$

Should the surface be sufficiently limp that it can support no normal stresses, P_n would vanish, so that this equation shows how a flexible surface can provide the reflexion of an image of exactly opposite strength, distorted only by second-order terms of the surface response. This result seems important because it is contrary to the concept that, if a surface is allowed to respond to turbulent flow, it will generate sound in a fundamentally more efficient manner than can turbulence in association with a rigid surface. A pressure-release surface appears to play a purely passive role, and it is of considerable interest to know whether or not surfaces of intermediate properties radiate sound more effectively. This is the problem that we deal with next, but, since the situation becomes somewhat more intricate, we neglect second-order terms in surface response together with viscous effects.

3. Surface response and the radiation field

The neglect of viscous effects and terms that are non-linear in the surface response velocity allows the sound field to be described by more tractable approximations of equations (2.7) and (2.8)

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+}[T_{ij}]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [p]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{d\mathbf{y}}{r}, \tag{3.1}$$

$$0 = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [p]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{\partial\mathbf{y}}{r}. \tag{3.2}$$

$n+$ is written to signify the outward normal to the surface $s+$, so that

$$\partial/\partial x_n \int_{s+} = \partial/\partial x_{n+} \int_s .$$

p is the pressure and $\bar{p}$ is the mean value of the fluid density ρ.

The surface responds to the pressure p with velocity v_n, so that p and v_n are related through the surface response equation. In general, whenever the surface is homogeneous, the response is described by a linear differential equation which we can write

$$p = F(v_n), \tag{3.3}$$

$F(v_n)$ being a collection of differential or integral operators, acting on the normal velocity v_n. It is an important feature of this theory that the operator F commutes in an interesting way. This can be shown as follows.

We write the integral involving surface pressure in terms of the surface velocity by using the response equation (3.3). The operator F can be written as

$$\Sigma_q A_q \frac{\partial^q}{\partial y_\alpha^q} + B_q \frac{\partial^q}{\partial t^q} + \Sigma_m C_{qm} \frac{\partial^m}{\partial y_\alpha^m} \frac{\partial^{q-m}}{\partial t^{q-m}},$$

where the co-ordinate y_α implies some direction lying in the surface s. Then

$$\begin{aligned}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [p]\frac{d\mathbf{y}}{r} &= \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [F(v_n)]\frac{d\mathbf{y}}{r}\\ &= \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s \left[\left\{\Sigma_q A_q \frac{\partial^q}{\partial y_\alpha^q} + B_q \frac{\partial^q}{\partial t^q} + \Sigma_m C_{qm} \frac{\partial^m}{\partial y_\alpha^m}\frac{\partial^{q-m}}{\partial t^{q-m}}\right\} v_n\right]\frac{d\mathbf{y}}{r},\end{aligned}$$

$$\begin{aligned}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [p]\frac{d\mathbf{y}}{r} = \Sigma_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s \left[A_q \frac{\partial^q v_n}{\partial y_\alpha^q}\right]\frac{d\mathbf{y}}{r}\\ + \Sigma_q B_q \frac{\partial^q}{\partial t^q}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}\\ + \Sigma_q \Sigma_m C_{qm}\frac{\partial^{q-m}}{\partial t^{q-m}}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s \left[\frac{\partial^m v_n}{\partial y_\alpha^m}\right]\frac{d\mathbf{y}}{r}.\end{aligned} \tag{3.4}$$

The first term on the right-hand side is now treated independently

$$\begin{aligned}\Sigma_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s A_q\left[\frac{\partial^q v_n}{\partial y_\alpha^q}\right]\left(\mathbf{y}, t-\frac{r}{a_0}\right)\frac{d\mathbf{y}}{r}\\ = \Sigma_q A_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s \left\{\frac{\partial}{\partial y_\alpha}\left[\frac{\partial^{q-1} v_n}{\partial y_\alpha^{q-1}}\right] + \frac{\partial}{\partial x_\alpha}\left[\frac{\partial^{q-1} v_n}{\partial y_\alpha^{q-1}}\right]\right\}\frac{d\mathbf{y}}{r}.\end{aligned} \tag{3.5}$$

This follows by writing $|\mathbf{x}-\mathbf{y}|$ for r, a step that also allows the right-hand side to be rewritten as

$$\Sigma_q A_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s \left\{\frac{\partial}{\partial y_\alpha} + \frac{\partial}{\partial x_\alpha}\right\}\left[\frac{1}{r}\frac{\partial^{q-1} v_n}{\partial y_\alpha^{q-1}}\right] d\mathbf{y}. \tag{3.6}$$

The derivative with respect to y_α disappears by integrating the expression in the y_α direction. Surface terms have already been assumed to vanish at large distances, so that equation (3.5) can be rewritten

$$\Sigma_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s A_q\left[\frac{\partial^q v_n}{\partial y_\alpha^q}\right]\frac{d\mathbf{y}}{r} = \Sigma_q A_q \frac{\partial}{\partial x_\alpha}\frac{1}{4x}\frac{\partial}{\partial x_{n+}}\int_s \left[\frac{\partial^{q-1} v_n}{\partial y_\alpha^{q-1}}\right]\frac{d\mathbf{y}}{r}.$$

This step can be repeated as often as is necessary to show how the differential operator commutes

$$\Sigma_q \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s A_q \left[\frac{\partial^q v_n}{\partial y_\alpha^q}\right]\frac{d\mathbf{y}}{r} = \Sigma_q A_q \frac{\partial^q}{\partial x_\alpha^q}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}. \tag{3.7}$$

Clearly, this property holds for the mixed operator in the third term on the right-hand side of equation (3.4), so that that equation can be rewritten

$$\begin{aligned}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [F(v_n)]\frac{d\mathbf{y}}{r} &= \left\{\Sigma_q A_q \frac{\partial^q}{\partial x_\alpha^q} + B_q \frac{\partial^q}{\partial t^q} + \Sigma_m C_{qm}\frac{\partial^m}{\partial x_\alpha^m}\frac{\partial^{q-m}}{\partial t^{q-m}}\right\}\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r} \\ &= F\left\{\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}\right\}.\end{aligned} \tag{3.8}$$

$\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}$ is the velocity perturbation measured in the direction of $n+$, induced by a pressure field

$$p_v = \frac{-1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{d\mathbf{y}}{r}.$$

This is readily demonstrated by applying the momentum equation in the source-free space. p_v is the pressure described by the surface-velocity term in equation (3.1), hence

$$\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r},$$

is the normal velocity perturbation that pressure induces. Equations (3.3) and (3.8) combine to show how the surface-pressure term is precisely equal to the pressure necessary to force the surface to respond with the normal velocity field associated with the pressure p_v in free flow. That pressure we shall denote by p_r

$$p_r = \frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [p]\frac{d\mathbf{y}}{r} = F\left\{\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}\right\}. \tag{3.9}$$

The radiation equations can then be rewritten in a convenient closed form. They comprise three simultaneous equations for the pressures $p(\mathbf{x},t)$, $p_v(\mathbf{x},t)$ and $p_r(\mathbf{x},t)$

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+}[T_{ij}]\frac{d\mathbf{y}}{r} + F\left\{\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}\right\} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{d\mathbf{y}}{r}, \tag{3.10}$$

$$= \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+}[T_{ij}]\frac{d\mathbf{y}}{r} + p_r + p_v, \tag{3.11}$$

$$0 = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r} - F\left\{\frac{1}{4\pi}\frac{\partial}{\partial x_{n+}}\int_s [v_n]\frac{d\mathbf{y}}{r}\right\} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{d\mathbf{y}}{r}, \tag{3.12}$$

$$= \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r} - p_r + p_v, \tag{3.13}$$

$$\bar{\rho}\frac{\partial p_r}{\partial t} = -F\left(\frac{\partial p_v}{\partial x_{n+}}\right). \tag{3.14}$$

Equations (3.13) and (3.14) are auxiliary equations which may be solved to yield the value of p_v for any particular surface property. For a rigid surface p_v is zero, so that

$$p_r = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V-}[T_{ij}]\frac{d\mathbf{y}}{r},$$

accounting for the reflexion property of rigid surfaces. When the surface is limp enough that it appears as a pressure-release condition, p_r is zero, and the surface reflects a negative image. Should the surface have properties identical to the fluid, p_v would equal p_r so that the image system would vanish.

Intermediate surface properties can be described by a surface impedance, but since this is a function highly sensitive to changes in frequency and scale, the equations have to be rewritten in spectral form. We do this by defining generalized Fourier transforms of the pressures that appear in equation (3.11), introducing asterisks to denote transformed variables.

$$p(\mathbf{x},t) = \iint p^*(x_n, \mathbf{k}, \omega)\, e^{i\omega t} e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}\, d\omega,$$

$$\frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V\pm}[T_{ij}]\frac{d\mathbf{y}}{r} = \iint T\pm(x_n, \mathbf{k}, \omega)\, e^{i\omega t} e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}\, d\omega, \qquad (3.15)$$

$$p_v(\mathbf{x},t) = \iint p_v^*(x_n, \mathbf{k}, \omega)\, e^{i\omega t} e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}\, d\omega, \qquad (3.16)$$

$$p_r(\mathbf{x},t) = \iint p_r^*(x_n, \mathbf{k}, \omega)\, e^{i\omega t} e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}\, d\omega, \qquad (3.17)$$

$\mathbf{k}$ is a two-dimensional wave vector and ω is the frequency.

The response function $p = F(v_{n+})$ becomes a simple algebraic factor in the transformed variables. That factor is known as the impedance which we denote by z, where

$$p^* = zv_{n+}^*. \qquad (3.18)$$

In passive systems, such as the surface considered here, the real part of the impedance must be positive. It is related to the direction of power flow, which cannot be of a type where the surface does work on the fluid. If there is no dissipation, the real part of z would be zero since there would then be no power flowing from the fluid to the surface. The imaginary part of the impedance can be either negative or positive, depending on whether the surface is excited above or below the free-wave frequency.

In the source free flow beyond the turbulence, the normal fluid velocity is also linearly related to the pressure, this time by a characteristic wave impedance, a quantity we denote by z_w. This property enters the problem when we define the Fourier transform of the particle velocity u_{n+}, induced by the pressure p_v:

$$\bar{\rho}\frac{\partial u_{n+}}{\partial t} = -\frac{\partial p_v}{\partial x_{n+}}, \qquad (3.19)$$

$$u_{n+} = \iint u_{n+}^*(x_n, \mathbf{k}, \omega)\, e^{i\omega t} e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}\, d\omega, \qquad (3.20)$$

$$p_v^* = -z_w u_{n+}^*. \qquad (3.21)$$

Again, since energy flows in the negative $n+$ direction above the turbulent flow, the real part of the wave impedance z_w must be greater than, or equal to, zero, The Fourier transform of equation (3.14) can now be expressed in a form where p_r^* is defined explicitly in terms of the impedance functions and the pressure p_v^*

$$p_r^* = zu_{n+}^* = -(z/z_w)\,p_v^*. \tag{3.22}$$

Equations (3.11) and (3.13), when written in terms of the transformed parameter, then yield an explicit expression for the spectral form of the radiated pressure:

$$0 = T_- + p_v^*(1+z/z_w), \tag{3.23}$$

$$p^* = T_+ + T_- - \frac{2T_-}{(z/z_w+1)},$$

$$p^* = T_+ + \frac{(z/z_w-1)}{(z/z_w+1)}T_-. \tag{3.24}$$

At first sight, this result appears as rather a surprise, for the function of the impedance ratio that multiplies T_- is nothing more than the reflexion coefficient familiar in many kinds of wave problems. It is a simple matter to demonstrate that for radiating waves that coefficient cannot exceed unity in absolute magnitude, so that surface sources account for an additional sound that is essentially weaker than that of the turbulence alone. It is a surprising result in view of the clear possibility that surface motion could induce powerful sources excited by the non-radiating turbulent pressure field, a source system potentially vastly more powerful than the volume quadrupoles T_+ or T_-, i.e. the surface might act as a sounding board. That this is not the case rests on one essential condition that, if violated, would completely change the character of the turbulence-induced sound. That is the proviso that the surface motion is completely described by a linear differential equation. Boundaries, supports and inhomogeneities of the surface structure are thereby excluded. Such structures are known to reflect plane waves, as determined by the reflexion coefficient. Since we have decomposed the radiation into a system of such waves by taking the Fourier transform, the result seems less surprising, and in fact becomes the obvious answer.

Again the three limiting conditions are evident. When the surface impedance is very high in comparison with the wave impedance, a perfect reflexion is established, so that $p^* = T_+ + T_-$. When the surface impedance is very much lower than the wave impedance, a perfect reflexion, of opposite strength is apparent, $p^* = T_+ - T_-$. However, when the surface impedance exactly equals the wave impedance, the image system vanishes, $p^* = T_+$. The directions of energy flow impose restrictions on the values of z and z_w that guarantee that z cannot be the exact negative of z_w, so that the possibility of a singularity in equation (3.24) as $(1+z/z_w)$ approaches zero is avoided.

Equation (3.24), when written in terms of the reflexion coefficient R is the basic result of this paper:

$$p^* = T_+ + RT_-, \tag{3.25}$$

where

$$R = (z-z_w)/(z+z_w). \tag{3.26}$$

This is an exact equation within the linear and inviscid boundary conditions, but it is in the distant radiation field that the result is likely to prove most useful. There the three-dimensional Fourier transform of pressure p^* is not the most significant parameter, and equation (3.25) should be inverted to yield an expression for the pressure at a particular point and at a particular frequency. The most straightforward way of doing this is to revert to equations (3.11), (3.13), and (3.14), and develop them again with that particular object in mind. However, the result described in equation (3.25) leads us to expect that surface response cannot fundamentally augment the sound power radiated by turbulence, and we shall see this more clearly in the section that follows.

4. The distant sound field of a locally turbulent region

It is possible, but by no means obvious, that a turbulent boundary layer formed on the skin of a large vehicle may appear to a distant observer as similar to turbulent flow established on a plane surface. This is the situation we examine now. We suppose that the region containing turbulent flow is small in comparison with the distance separating the turbulence from a distant observer. We also suppose that any flexural surface motion is confined to the vicinity of the turbulence, as it is bound to be whenever the surface material is dissipative. Under these conditions, we may disregard all terms in equations (3.10) to (3.14) that fall off with distance more quickly than $1/r$. Furthermore, we may regard $\partial r/\partial x_i$ as a directional constant which we will denote by

$$\beta_i = \partial r/\partial x_i. \tag{4.1}$$

The main analytical simplification that this step induces is that derivatives of the pressure p_v with respect to the field point $\mathbf{x}$ act only on the time, a familiar aspect of Lighthill's aerodynamic noise theory

$$\frac{\partial p_v}{\partial x_i} = -\frac{\beta_i}{a_0}\frac{\partial p_v}{\partial t}. \tag{4.2}$$

This far-field property allows the response operator F, as it appears in equation (3.8), to be rewritten as a set of time derivatives only

$$F = \sum_q \left\{A_q \frac{\partial^q}{\partial x_\alpha^q} + B_q \frac{\partial^q}{\partial t^q} + \sum_m C_{qm} \frac{\partial^m}{\partial x_\alpha^m}\frac{\partial^{q-m}}{\partial t^{q-m}}\right\},$$

$$F = \sum_q \left\{(-1)^q A_q \frac{\beta_\alpha^q}{a_0^q} + B_q + \sum_m (-1)^m C_{qm} \frac{\beta_\alpha^m}{a_0^m}\right\}\frac{\partial^q}{\partial t^q}. \tag{4.3}$$

Again, since the structural response varies with changes in frequency, we can advance more rapidly by conducting a spectral decomposition of the radiation field into its frequency components. However, in this far-field situation there is no need to Fourier synthesize the spatial variation, as was necessary in the preceding section. Once more, the differential operator F becomes a simple algebraic factor in the transformed variables. In fact, that factor is nothing more than the impedance z that we have already defined, with the wave vector set equal to a particular function of frequency

$$\mathbf{k}_\alpha = -\frac{\omega}{a_0}\beta_\alpha. \tag{4.4}$$

This relation is that found in a plane acoustic wave travelling from the turbulent source in a direction determined by the direction cosines β_α. The particular value of z for the plane wave condition of equation (4.4) is the specific acoustic impedance of the surface, a quantity usually denoted by z_s (e.g. Morse 1948). For a particular homogeneous structure, z_s is a function of direction and frequency only. In a similar way equation (3.19) can be rewritten for the far-field case

$$\bar{\rho}\frac{\partial u_{n+}}{\partial t} = -\frac{\partial p_v}{\partial x_{n+}} = \frac{\beta_{n+}}{a_0}\frac{\partial p_v}{\partial t}. \tag{4.5}$$

Then the spectral component of normal velocity is related to the pressure through the acoustic wave impedance, the particular value of z_w when

$$k_{n+} = -(\omega/a_0)\,\beta_{n+},$$

a value we denote by z_{ws}.

The far-field analysis at a particular frequency is then identical to that of the preceding section. We define Fourier transforms of the pressures that feature in the radiation equations, but only the temporal parameter is decomposed, the asterisk again denoting a Fourier transform. Equations (3.11), (3.13), and (3.14) then become

$$p^*(\mathbf{x},\omega) = T_+(\mathbf{x},\omega) + p_r^*(\mathbf{x},\omega) + p_v^*(\mathbf{x},\omega), \tag{4.6}$$

$$0 = T_-(\mathbf{x},\omega) - p_r^*(\mathbf{x},\omega) + p_v^*(\mathbf{x},\omega), \tag{4.7}$$

$$p_r^*(\mathbf{x},\omega) = -\frac{z_s}{z_{ws}}\,p_v^*(\mathbf{x},\omega). \tag{4.8}$$

The pressures p_r^* and p_v^* can be eliminated from these equations to show how the sound radiated to a distant point $\mathbf{x}$, at frequency ω, is determined by the free turbulence and the surface-reflexion coefficient for *plane acoustic waves* R_a,

$$R_a = (z_s - z_{ws})/(z_s + z_{ws}), \tag{4.9}$$

$$p^*(\mathbf{x},\omega) = T_+(\mathbf{x},\omega) + R_a T_-(\mathbf{x},\omega). \tag{4.10}$$

The acoustic reflexion coefficient R_a cannot exceed unity in absolute magnitude, so that the sound radiated to large distances from the flow can never exceed that due to the sum of the real and image flow turbulence with some phase difference determined by the surface response. The non-propagating near pressure field is thus unable to use the homogeneous surface as a sounding board to augment the total radiation in any fundamentally more efficient way. The sound remains quadrupole and increases with the eighth power of a typical flow velocity, a result that is in sharp contrast with the usual argument that surface motion induces equivalent simple sources and a sound increasing with the fourth power of velocity. More complex structures may indeed behave in that way but plane homogeneous surfaces play an essentially passive role in the problem of turbulent-boundary-layer noise.

5. Conclusion

The main results of the paper are that the influence of a large, plane, homogeneous boundary supporting turbulent flow is simply to 'reflect' the quadrupole sound generated by the turbulence. The reflexion coefficient is real for propagating waves but may have positive or negative values depending on the surface properties. Consequently, the presence of the surface does not fundamentally increase the efficiency of acoustic radiation by turbulent flow. For non-propagating components of the pressure field, the reflexion coefficient may be complex, and an increase in the near-field pressure may result from surface motion.

In applying this result to the boundary-layer case, it is pertinent to consider how relevant the large, plane, homogeneous boundary can be to practical problems. Stiffened, or locally supported structures, do not fall within this category, and these conclusions may prove quite misleading in some cases of practical interest. However, should homogeneous panels be large enough to support free waves devoid of significant modal structure, the vibration field could be regarded as occurring in an infinite panel without incurring appreciable error. To comply with this condition, the product of the loss factor with panel area should greatly exceed that of perimeter with wavelength, so that waves decay within a distance smaller than the typical panel scale and are not reflected to form a standing-wave, or modal, component.

Even though the possibility of surface motion implies no fundamentally more efficient method of sound production, it is likely to lead to a slightly greater acoustic output from a turbulent boundary layer. This is because the most powerful sources, the lateral quadrupoles associated with the high mean-velocity gradient (Lighthill 1954), are opposed by their images when the reflexion is complete, as it is in the case of a rigid surface. As the reflexion is changed, in either phase or magnitude, the cancellation becomes less complete, and these relatively powerful sources soon make themselves felt on the radiation field. However, it is unlikely that any practical situation should arise in which quadrupole radiation plays a crucial role. There would inevitably be other, more powerful, sources induced by either small-panel motion, or dipole systems near surface discontinuities, that would mask the quadrupoles of the turbulent volume—but that is really the main point of this paper. The possibility of powerful sources being associated with plane homogeneous surfaces is shown to be without a rigorous foundation and that the emphasis in flow-noise studies should be put elsewhere, where sources of fundamentally higher efficiency are to be found.

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